

# A spectral stability theorem for large forbidden subgraphs

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## Abstract

Let  $\mu(G)$  be the largest eigenvalue of a graph  $G$ , let  $K_r(s_1, \dots, s_r)$  be the complete  $r$ -partite graph with parts of size  $s_1, \dots, s_r$ , and let  $T_r(n)$  be the  $r$ -partite Turán graph of order  $n$ . Our main result is:

For all  $r \geq 2$  and all sufficiently small  $c > 0$ ,  $\varepsilon > 0$ , every graph  $G$  of sufficiently large order  $n$  with  $\mu(G) \geq (1 - 1/r - \varepsilon)n$  satisfies one of the conditions:

- (a)  $G$  contains a  $K_{r+1}(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$ ;
- (b)  $G$  differs from  $T_r(n)$  in fewer than  $(\varepsilon^{1/4} + c^{1/(8r+8)})n^2$  edges.

In particular, this result strengthens the stability theorem of Erdős and Simonovits.

**Keywords:** *stability, forbidden subgraphs,  $r$ -partite subgraphs; largest eigenvalue of a graph; spectral Turán theorem.*

This note is part of an ongoing project aiming to build extremal graph theory on spectral grounds, see, e.g., [3] and [7, 14].

Let  $\mu(G)$  be the largest adjacency eigenvalue of a graph  $G$ , let  $K_r(s_1, \dots, s_r)$  be the complete  $r$ -partite graph with parts of size  $s_1, \dots, s_r$ , and let  $T_r(n)$  be the  $r$ -partite Turán graph of order  $n$ . In [6] we extended the Erdős-Simonovits stability theorem [4], [15] as:

*Let  $r \geq 2$ ,  $1/\ln n < c < r^{-3(r+14)(r+1)}$ ,  $0 < \varepsilon < r^{-24}$ , and  $G$  be a graph of order  $n$ . If  $G$  has  $\lceil (1 - 1/r - \varepsilon)n^2/2 \rceil$  edges, then  $G$  satisfies one of the conditions:*

- (a)  $G$  contains a  $K_{r+1}(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$ ;
- (b)  $G$  differs from  $T_r(n)$  in fewer than  $(\varepsilon^{1/3} + c^{1/(3r+3)})n^2$  edges.

Here we derive essentially the same conclusion from the weaker premise  $\mu(G) > (1 - 1/r - \varepsilon)n$ :

**Theorem 1** *Let  $r \geq 2$ ,  $1/\ln n < c < r^{-8(r+21)(r+1)}$ ,  $0 < \varepsilon < 2^{-36}r^{-24}$ , and  $G$  be a graph of order  $n$ . If  $\mu(G) > (1 - 1/r - \varepsilon)n$ , then  $G$  satisfies one of the conditions:*

- (a)  $G$  contains a  $K_{r+1}(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$ ;
- (b)  $G$  differs from  $T_r(n)$  in fewer than  $(\varepsilon^{1/4} + c^{1/(8r+8)})n^2$  edges.

## Remarks

- Since  $\mu(G)$  is at least the average degree of  $G$ , Theorem 1 implies essentially the above extension of the Erdős-Simonovits stability theorem.
- The relation between  $c$  and  $n$  in Theorem 1 needs explanation. First, for fixed  $c$ , it shows how large must be  $n$  to get a valid conclusion. But, in fact, the relation is subtler, for  $c$  itself may depend on  $n$ , e.g., letting  $c = 1/\ln \ln n$ , the conclusion is meaningful for sufficiently large  $n$ .
- Choosing randomly a graph of order  $n$  with  $\lceil (1 - 1/r)n^2/2 \rceil$  edges, we can find a graph containing no  $K_2(\lfloor c' \ln n \rfloor, \lfloor c' \ln n \rfloor)$  and differing from  $T_r(n)$  in more than  $c''n^2$  edges for some positive  $c'$  and  $c''$ , independent of  $n$ . Hence, condition (a) is essentially best possible.
- The factor  $\varepsilon^{1/4} + c^{1/(8r+8)}$  in condition (b) is far from the best one, but is simple.

To prove Theorem 1, we introduce two supporting results. Our notation follows [1]; given a graph  $G$ , we write:

- $|G|$  for the number of vertices set of  $G$ ;
- $e(G)$  for the number of edges of  $G$ ;
- $\delta(G)$  for the minimum degree of  $G$ ;
- $k_r(G)$  for the number of  $r$ -cliques of  $G$ .

An  $r$ -joint of size  $t$  is the union of  $t$  distinct  $r$ -cliques sharing an edge. We write  $js_r(G)$  for the maximum size of an  $r$ -joint in a graph  $G$ .

The following two facts play crucial roles in our proof.

**Fact 2 ([14], Theorem 4)** *Let  $r \geq 2$ ,  $0 < b < 2^{-10}r^{-6}$ ,  $n \geq r^{20}$ , and  $G$  be a graph of order  $n$ . If  $\mu(G) > (1 - 1/r - b)n$ , then  $G$  satisfies one of the conditions:*

- (i)  $js_{r+1}(G) > n^{r-1}/r^{2r+5}$ ;
- (ii)  $G$  contains an induced  $r$ -partite subgraph  $G_0$  satisfying  $|G_0| \geq (1 - 4b^{1/3})n$  and  $\delta(G_0) > (1 - 1/r - 7b^{1/3})n$ .

**Fact 3 ([5], Theorem 1)** *Let  $r \geq 2$ ,  $c^r \ln n \geq 1$ , and  $G$  be a graph of order  $n$ . If  $k_r(G) \geq cn^r$ , then  $G$  contains a  $K_r(s, \dots, s, t)$  with  $s = \lfloor c^r \ln n \rfloor$  and  $t > n^{1-c^{r-1}}$ .  $\square$*

**Proof of Theorem 1** Let  $G$  be a graph of order  $n$  with  $\mu(G) > (1 - 1/r - \varepsilon)n$ . Define the procedure  $\mathcal{P}$  as follows:

**While**  $js_{r+1}(G) > n^{r-1}/r^{2r+5}$  **do**

*Select an edge contained in  $\lceil n^{r-1}/r^{2r+5} \rceil$  cliques of order  $r+1$  and remove it from  $G$ .*

Set for short  $\theta = c^{1/(r+1)}r^{2r+5}$  and assume first that  $\mathcal{P}$  removes at least  $\lceil \theta n^2 \rceil$  edges before stopping. Then

$$k_{r+1}(G) \geq \theta n^{r-1}/r^{2r+5} = c^{1/(r+1)}n^{r+1},$$

and Fact 3 implies that  $K_{r+1}(\lfloor c \ln n \rfloor, \dots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil) \subset G$ . Thus condition (a) holds, completing the proof.

Assume now that  $\mathcal{P}$  removes fewer than  $\lceil \theta n^2 \rceil$  edges before stopping; write  $G'$  for the resulting graph.

Letting  $\mu(X)$  be the largest eigenvalue of a Hermitian matrix  $X$ , recall Weyl's inequality

$$\mu(B) \geq \mu(A) - \mu(A - B),$$

holding for any Hermitian matrices  $A$  and  $B$ . Also, recall that  $\mu(H) \leq \sqrt{2e(H)}$  for any graph  $H$ . Applying these results to the graphs  $G$  and  $G'$ , we find that

$$\mu(G') \geq \mu(G) - \sqrt{2\theta}n \geq \left(1 - 1/r - \varepsilon - \sqrt{2\theta}\right)n.$$

From  $\ln n \geq 1/c \geq r^{8(r+21)(r+1)}$  we easily get  $n > r^{20}$ . Set for short  $a = \left(\varepsilon + \sqrt{2\theta}\right)^{1/3}$ . Since

$$\varepsilon + \sqrt{2\theta} \leq 2^{-36}r^{-24} + 2r^{-4(r+21)(r+1)} < 2^{-10}r^{-6},$$

and  $js_{r+1}(G') \leq n^{r-1}/r^{2r+5}$ , Fact 2 implies that  $G'$  contains an induced  $r$ -partite subgraph  $G_0$ , satisfying  $|G_0| \geq (1 - 4a)n$  and  $\delta(G_0) > (1 - 1/r - 7a)n$ .

Let  $V_1, \dots, V_r$  be the parts of  $G_0$ . For every  $i \in [r]$ , we see that

$$|V_i| \geq n - \sum_{s \in [r] \setminus \{i\}} |V_s| \geq n - (r-1)(n - \delta(G_0)) \geq (1/r - 7(r-1)a)n.$$

For each  $i \in [r]$ , select a set  $U_i \subset V_i$  with

$$|U_i| = \lceil (1/r - 7(r-1)a)n \rceil,$$

and write  $G_1$  for the graph induced by  $\cup_{i=1}^r U_i$ . Clearly  $G_1$  can be made complete  $r$ -partite by adding at most

$$((1 - 1/r)|G_1| - \delta(G_1))|G_1|/2$$

edges. We see that

$$\delta(G_1) \geq \delta(G_0) - |G_0| + |G_1| \geq -n/r - 4an + |G_1|,$$

and so,

$$\begin{aligned} (1 - 1/r)|G_1| - \delta(G_1) &\leq (1/r + 4a)n - |G_1|/r \\ &= (1/r + 4a)n - (1/r - 7(r-1)a)n = 7ran. \end{aligned}$$

Therefore,  $G_1$  can be made complete  $r$ -partite by adding at most

$$7ran|G_1|/2 < 4ran^2$$

edges.

The complete  $r$ -partite graph with parts  $U_1, \dots, U_r$  can be transformed into  $T_r(n)$  by changing at most  $(n - |G_1|)n$  edges. Since

$$(n - |G_1|)n \leq (n - r(1/r - 7(r-1)a)n)n = 7r(r-1)an^2,$$

we find that  $G$  differs from  $T_r(n)$  in at most  $(\theta + (7r^2 - 3r)a)n^2$  edges. Now, condition (b) follows in view of

$$\begin{aligned} \theta + (7r^2 - 3r)a &= \theta + (7r^2 - 3r)\left(\varepsilon + \sqrt{2\theta}\right)^{1/3} < \theta + 8r^2\varepsilon^{1/3} + 2(7r^2 - 3r)\theta^{1/6} \\ &< 8r^2\varepsilon^{1/3} + r^6\theta^{1/6} < \varepsilon^{1/4} + r^{(2r+5)/6+6}c^{1/(6r+6)} \leq \varepsilon^{1/4} + c^{1/(8r+8)}. \end{aligned}$$

The proof is completed. □

### Concluding remark

Finally, a word about the project mentioned in the introduction: in this project we aim to give wide-range results that can be used further, adding more integrity to spectral extremal graph theory.

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